Published by Institute of Physics Publishing for SISSA/ISAS

Received: *July 9, 2002*
Accepted: *September 13, 2002*

Kaluza-Klein bundles and manifolds of exceptional holonomy

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Abstract: We show how in the presence of RR two-form field strength the conditions for preserving supersymmetry on six- and seven-dimensional manifolds lead to certain generalizations of monopole equations. For six dimensions the string frame metric is Kähler with the complex structure that descends from the octonions if in addition we assume $F^{(1,1)} = 0$. The susy generator is a gauge covariantly constant spinor. For seven dimensions the string frame metric is conformal to a G_2 metric if in addition we assume the field strength to obey a self-duality constraint. Solutions to these equations lift to geometries of G_2 and Spin(7) holonomy respectively.

Keywords: M-Theory, D-branes, Superstring Vacua.

JHEP09(2002)033

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1. Introduction

Recent progress in understanding $\mathcal{N} = 1$ dynamics has to a large extent relied on exploiting dual realizations of such theories. A prominent example of such dual pairs are D6 branes of type IIA string theory wrapping supersymmetric three cycles in a Calabi-Yau threefold on one side and M-theory compactifications on manifolds of G_2 holonomy on the other side, [1, 2]. A less studied example realizing such a duality for $\mathcal{N} = 1$ theories in three dimensions are D6 branes wrapping supersymmetric four cycles in a G_2 manifold and M-theory compactifications on manifolds of Spin(7) holonomy. Together with D6-branes in flat space, this exhausts the list of spacetime filling D6-branes partially wrapped on supersymmetric cycles of an internal d-dimensional space X that lift in M-theory to compactifications on $(d+1)$-dimensional spaces Y preserving half as many supersymmetries as does X (see table 1).

In all these cases, the power of the M-theory realizations resides in the fact that they are completely geometrical, no background fluxes being switched on. Much progress has been achieved recently in the study of these M-theory constructions as well as in the derivation of explicit metrics for certain non-compact manifolds of exceptional holonomy (for a review with an extensive list of references see e.g. [3]).

	Hol(X)	susy cycle	Hol(Y)
$d = 3$	$\{1\}$	$\{pt\}$	SU(2)
$d = 6$	SU(3)	SLAG	G_2
$d = 7$	G_2	coassociative	Spin(7)

Table 1: Supersymmetric cycles and lifts to M-theory.

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Our purpose here is a further clarification of the connections between the geometries and associated structures in d and $d+1$ dimensions, as well as the general properties of the Kaluza-Klein (monopole) bundles. The strategy is rather conventional — we will analyze the conditions for preserving supersymmetry in type IIA string theory with nontrivial RR vector field and dilaton, namely all the fields coming from the eleven-dimensional metric. The result is a set of gauge equations in six and seven dimensions which are close relatives to the familiar three-dimensional monopole equations that arise in the flat case; schematically they take the form

$$\partial_a \phi \sim \eta_{abc} F^{bc} \,, \tag{1.1}$$

where in six dimensions the indices are holomorphic and the form η is only of type $(3,0)$, whereas in seven dimensions the indices span all of the seven dimensions and the three-form is the G_2 form Φ.

For D6-branes on special lagrangian three-cycles in a non-compact Calabi-Yau threefold, we also show that the almost complex structure descending from the octonions is integrable and that the ten-dimensional string frame metric in the presence of the D6-branes together with this complex structure makes the internal space into a Kähler manifold, if the further constraint $F^{(1,1)} = 0$ is imposed, which is stronger than the condition $F^{ab} J_{ab} = 0$, required by supersymmetry. The ten-dimensional string frame metric is a warped product whose internal part is this Kähler metric. The supersymmetry generator, however, is then a *gauge covariantly constant* spinor. In seven dimensions, by imposing the monopole equations we find that, in the presence of the D6-branes, the string frame metric on the internal manifold is conformal to a metric of G_2 holonomy, if in addition a self-duality of the field strength is assumed.

The structure of the paper is as follows. In section 2 we review the case of D6-branes in flat space. In section 3 we then turn to supersymmetric D6-branes on Calabi-Yau manifolds where we present three alternative derivations of the monopole equations and the geometry in the string frame metric. First we use the supersymmetry constraints in type IIA in the presence of nontrivial dilaton and RR two-form field strength. Then we derive the same result from the existence of a G_2 structure on the lift (see [4] for a similar ansatz) and lastly from the existence of a self-dual spin connection on the lift [5]. In section 4 the same line of thought as in section 3 is applied to a background of D6-branes on supersymmetric four-cycles in G_2 manifolds that lift to Spin(7) manifolds. In section 5 we summarize and discuss our results.

2. Review of D6-branes in flat space, or: $\mathrm{SU}(2) \to \{1\}$

Before we start to analyze supersymmetric D6-branes in Calabi-Yau manifolds, let us recall the well-understood case of N coincident D6-branes of type IIA in flat ten-dimensional space. These branes are magnetically charged under the RR one-form potential A. This potential can be identified with the connection of the principal U(1)-bundle over $\mathbb{R}^3 - \{0\}$, the transverse space to the D6-branes, describing a magnetic monopole of charge N. The

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associated Maxwell equations imply that

$$\partial_i V(x) = -\frac{1}{2}\epsilon_{ijk}F^{jk}(x) \quad \Leftrightarrow \quad dV = -\tilde{*}F, \qquad \text{where} \quad V(x) = \epsilon + \frac{N}{4\pi|\,x\,|} \tag{2.1}$$

with $F = dA$, an arbitrary integration constant ϵ and where the Hodge * is taken w.r.t. the flat metric on $\mathbb{R}^3$. The scalar potential $V(x)$ is harmonic on $\mathbb{R}^3 - \{0\}$ with $\Delta V(x) = -\frac{1}{2}\epsilon_{ijk}\partial^i F^{jk}(x) = -N\delta(x)$, where the δ-function indicates the presence of the monopoles. Solutions for the potential A are given by the well-known Wu-Yang monopole potentials.

When lifted to eleven dimensions, the configuration becomes purely geometrical and the magnetic monopole in three dimensions a gravitational instanton in four dimensions. The additional M-theory circle is loosely speaking the fiber of the monopole bundle. To be more precise, the four-dimensional transverse space has a metric [6, 7]

$$ds_4^2 = V(x)d\tilde{s}_3^2 + V^{-1}(x)(dz + A)^2\,, \tag{2.2}$$

where z is a periodic coordinate and $d\tilde{s}_3^2$ the euclidean metric on $\mathbb{R}^3$. The monopole equation (2.1) is precisely the requirement of anti-self-dual spin connection of the metric (2.2), which implies Ricci-flatness. The metric is an example of Hawking's multi-center metrics for gravitational instantons in a limit where the N centers coincide. The non-negative integration constant ϵ controls the asymptotic behavior of the circle parameterized by z. For $\epsilon = 0$ it decompactifies and the metric becomes asymptotically locally euclidean. For $\epsilon = 0$ and $N = 2$ it is the Eguchi-Hanson metric. For nonvanishing ϵ the circle remains compact also asymptotically. For $\epsilon = 1$ and $N = 1$ it is the Taub-NUT metric.

The background metric for the M-theory compactification is thus

$$ds_{11}^2 = e^{-2\alpha\phi}ds_{10}^2 + e^{2\beta\phi}(dz + A)^2 = d\tilde{s}_7^2 + ds_4^2\,, \tag{2.3}$$

where $d\tilde{s}_7^2$ is a flat Minkowski metric in seven dimensions, ds_4^2 is given in (2.2), $V = e^{-2\beta\phi}$ and where ds_{10}^2 is the physical metric in ten dimensions. The parameters α and β determine the frame in which this metric is given. For a string frame in ten dimensions they are $(\alpha, \beta) = (1/3, 2/3)$, but for comparability with different frames we will leave them general in most of the equations unless we refer explicitly to the ten-dimensional string frame. The physical ten-dimensional metric is thus a warped product

$$ds_{10}^2 = e^{2\alpha\phi}d\tilde{s}_7^2 + e^{2(\alpha-\beta)\phi}d\tilde{s}_3^2 \tag{2.4}$$

of two flat metrics. I.e. the presence of the D6-branes shows up in the string frame metric as a warping by $V^{-1/2}$ and $V^{1/2}$ of the longitudinal and transverse directions, respectively. Using the string frame relation $\beta = 2\alpha$, the monopole equation (2.1) can be rewritten as

$$dV = d\left(e^{-2\beta\phi}\right) = -*\left(e^{-\alpha\phi}F\right), \tag{2.5}$$

where the Hodge * is now taken w.r.t. the string frame metric $e^{-2\alpha\phi}d\tilde{s}_3^2$ on the internal space.

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3. $G_2 \to SU(3)$

In this section we will analyze the geometry of D6-branes wrapped on supersymmetric cycles in non-compact Calabi-Yau manifolds. As mentioned in the introduction, we start from the examination of the conditions for preserving supersymmetry in type IIA string theory in the presence of the RR vector field and a nontrivial dilaton. It is easiest to derive these conditions from reduction of the supersymmetry conditions of eleven-dimensional supergravity. To this end our conventions are as follows. The eleven-dimensional metric

$$ds_{11}^2 = e^{-2\alpha\phi} ds_{10}^2 + e^{2\beta\phi}(dz + A)^2 = d\tilde{s}_4^2 + ds_7^2 \tag{3.1}$$

is assumed to be the direct product of a Minkowski metric $d\tilde{s}_4^2$ with a non-compact G_2 metric ds_7^2, the latter having a U(1) isometry parameterized by the coordinate z. We assume that the field strength of the KK-vector A has nonvanishing components only in the internal dimensions. Also the dilaton ϕ depends nontrivially only on the internal coordinates. The metric ds_{10}^2 is the physical metric in ten dimensions; the parameters α and β determine the frame, with $(\alpha, \beta) = (1/3, 2/3)$ for the string frame in ten dimensions. The physical ten-dimensional metric is thus of a warped type

$$ds_{10}^2 = e^{2\alpha\phi} d\tilde{s}_4^2 + ds_6^2 \tag{3.2}$$

and the G_2 metric reads

$$ds_7^2 = e^{-2\alpha\phi} ds_6^2 + e^{2\beta\phi}(dz + A)^2 \,. \tag{3.3}$$

We will denote objects referring to the metric ds_7^2 with hats, whereas the others refer to the metric ds_6^2. Upper case frame indices run over the range $A, B, C = 1, \ldots, 7$, where the index 7 refers to the z-direction, lower case frame indices have the range $a, b, c = 1, \ldots, 6$.

3.1 Supersymmetry and holomorphic monopoles

With our assumptions on the eleven-dimensional metric (3.1) and the field strength being internal, the condition for $\mathcal{N} = 1$ supersymmetry in four dimensions reduces to the condition of having exactly one covariantly constant Majorana spinor on the internal seven-manifold with metric (3.3),

$$\hat{D}_A \epsilon = \left(\hat{\partial}_A + \frac{1}{4} \hat{\Gamma}_{BAC} \gamma^{BC} \right) \epsilon = 0 \,. \tag{3.4}$$

Using the following relations between the spin connection coefficients corresponding to the metrics ds_7^2 and ds_6^2,

$$\begin{aligned}
\hat{\Gamma}_{abc} &= e^{\alpha\phi} \left\{ \Gamma_{abc} + \alpha \left[\delta_{bc}(\partial_a \phi) - \delta_{ba}(\partial_c \phi) \right] \right\} \,, \\
\hat{\Gamma}_{azc} &= -\frac{1}{2} e^{(2\alpha+\beta)\phi} F_{ac} \,, \\
\hat{\Gamma}_{zbc} &= -\frac{1}{2} e^{(2\alpha+\beta)\phi} F_{bc} \,, \\
\hat{\Gamma}_{zzc} &= \beta e^{\alpha\phi} (\partial_c \phi) \,,
\end{aligned} \tag{3.5}$$

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together with the constraint that ϵ does not depend on z, the supersymmetry condition (3.4) reduces to the following system in six dimensions

$$e^{\alpha\phi}\left(D_a + \frac{1}{2}\alpha(\partial_b\phi)\,\gamma^b{}_a + \frac{1}{4}\check{F}_{ab}\gamma^b\gamma\right)\epsilon = 0\,, \tag{3.6}$$

$$-\frac{1}{2}e^{\alpha\phi}\left(\frac{1}{4}\check{F}_{ab}\gamma^{ab} + \beta\,(\partial_a\phi)\,\gamma^a\gamma\right)\epsilon = 0\,, \tag{3.7}$$

where we have defined

$$D_a = \partial_a + \frac{1}{4}\Gamma_{bac}\gamma^{bc}\,, \qquad \gamma = \gamma^7\,, \qquad \text{and} \quad \check{F}_{ab} = e^{(\alpha+\beta)\phi}F_{ab}\,.$$

We can now turn to possible solutions to (3.6), (3.7). We are interested in six-dimensional geometries that are related via Kaluza-Klein reduction to non-compact G_2 manifolds with metric (3.3). The spinor ϵ is then the covariantly constant Majorana spinor on the G_2 manifold, satisfying the identity [8]

$$\gamma_{AB}\,\epsilon = i\,\Phi_{ABC}\,\gamma^C\epsilon\,, \tag{3.8}$$

where Φ_{ABC} are the structure constants of the imaginary octonions. Defining $\psi_{abc} \equiv \Phi_{abc}$ and $J_{ab} \equiv \Phi_{ab7}$, this identity reduces from the six-dimensional perspective to

$$\gamma_{ab}\epsilon = i\,\psi_{abc}\gamma^c\epsilon + i\,J_{ab}\gamma\,\epsilon\,, \qquad \gamma_a\gamma\,\epsilon = -i\,J_{ab}\gamma^b\epsilon\,, \tag{3.9}$$

where in addition

$$J_a{}^b J_b{}^c = -\delta_a{}^c\,. \tag{3.10}$$

The last identity implies that $J_a{}^b$ defines an almost complex structure in six dimensions. Due to its antisymmetry, the riemannian metric is actually hermitian w.r.t. $J_a{}^b$.

Plugging (3.9) into equations (3.6), (3.7), we get

$$\left(D_a + \frac{i}{2}\alpha\,(\partial_b\phi)J^b_{\ a}\,\gamma\right)\epsilon + i\,\left(\frac{1}{2}\alpha\,(\partial_b\phi)\,\psi^b{}_{ac} - \frac{1}{4}\check{F}_{ab}J^b_{\ c}\right)\gamma^c\epsilon = 0\,, \tag{3.11}$$

$$\left(\frac{i}{4}\check{F}_{ab}J^{ab}\right)\gamma\epsilon + \left(\frac{i}{4}\check{F}_{ab}\psi^{ab}{}_c - i\,\beta\,(\partial_a\phi)J^a_{\ c}\right)\gamma^c\epsilon = 0\,. \tag{3.12}$$

Here we have dropped the overall factors in front of (3.6), (3.7); this means that all the equations we get afterwards have to be satisfied outside the locus in which $\phi = -\infty$.

Since the spinors $\gamma^A\epsilon$ are linearly independent for $A = 1, \ldots, 7$, [8], we can conclude that each of the two brackets in (3.12) has to vanish,

$$F^{ab}J_{ab} = 0\,, \tag{3.13}$$

$$\beta(\partial_a\phi)J^a_{\ c} = \frac{1}{4}\check{F}^{ab}\psi_{abc} \qquad \Leftrightarrow \qquad d\left(e^{-2\beta\phi}\right) = -*\left(e^{-\alpha\phi}\psi_3\wedge F\right). \tag{3.14}$$

Here we have defined the three-form $\psi_3 = \frac{1}{3!}\psi_{abc}e^a e^b e^c$ using the orthonormal (w.r.t. ds_6^2) cotangent frame e^a and the Hodge * is taken w.r.t. the string frame metric ds_6^2 on the internal space. The equivalence in (3.14) uses the multiplication properties of the octonionic

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structure constants (a collection of which can e.g. be found in the appendix of [5], whose conventions we follow). Using these identities one easily shows that, in the string frame $\beta = 2\alpha$, (3.14) reduces the gravitino equation (3.11) to

$$\left(D_a + \frac{i}{2}\alpha\,(\partial_b\phi)J^b{}_a\,\gamma - \frac{i}{8}\left[\check{F}_{ab}J^b{}_c + \check{F}_{cb}J^b{}_a\right]\gamma^c\right)\epsilon = 0\,. \tag{3.15}$$

Splitting the (co)tangent bundle into a holomorphic and an anti-holomorphic one w.r.t. $J_a{}^b$, one sees that only the $(1,1)$ part of F contributes to the square bracket in (3.15). Before moving on to analyze this equation, we note that by taking the dilaton to be constant, in addition to (3.13) we get $F^{(2,0)} = 0$, thus yielding hermitean YM equations. The six-dimensional structures and the role of $F^{(1,1)}$ in such a case are studied in [9].

Moreover it is also the $(1,1)$ part of F that controls the non-integrability of the almost complex structure. Using its representation $J_a{}^b = -i\epsilon^\dagger\gamma_a{}^b\gamma\epsilon$ and the gravitino equation, it follows that the Nijenhuis tensor takes the form

$$N^a_{bc} = \frac{1}{2}\left[J_c{}^d\check{F}_{de}\psi^e{}_b{}^a - J_b{}^d\check{F}_{de}\psi^e{}_c{}^a - \check{F}_{bd}(*\psi)^d{}_c{}^a + \check{F}_{cd}(*\psi)^d{}_b{}^a\right], \tag{3.16}$$

where $(*\psi)_{abc} = \frac{1}{3!}\epsilon_{defabc}\psi^{def}$ are the components of $*\psi_3$. In the holomorphic/antiholomorphic basis, labeled by $(a,\bar{a})$, we can use that $J_a{}^b = i\,\delta_a{}^b$ and that $\psi_{abc} = \frac{1}{2}\epsilon_{abc}$, $\psi_{\bar{a}\bar{b}\bar{c}} = \frac{1}{2}\epsilon_{\bar{a}\bar{b}\bar{c}}$, (see (3.24) and the remark following it), to find that the only nonvanishing components of the Nijenhuis tensor are

$$N^{\bar{a}}_{bc} = \frac{i}{2}\left(\check{F}_{c\bar{d}}\,\epsilon^{\bar{d}}{}_b{}^{\bar{a}} - \check{F}_{b\bar{d}}\,\epsilon^{\bar{d}}{}_c{}^{\bar{a}}\right) \qquad \text{and} \qquad N^a_{\bar{b}\bar{c}} = -\frac{i}{2}\left(\check{F}_{\bar{c}d}\,\epsilon^d{}_{\bar{b}}{}^a - \check{F}_{\bar{b}d}\,\epsilon^d{}_{\bar{c}}{}^a\right). \tag{3.17}$$

The almost complex structure that descends from the octonions is thus integrable if and only if

$$F^{(1,1)} = 0\,, \tag{3.18}$$

which is of course stronger than (3.13). In equation (3.23) we will actually show that the string frame metric ds_6^2 is then a Kähler metric w.r.t. to this complex structure.

In the rest of the paper we will assume the stronger condition (3.18) to hold. Together with (3.14) this reduces the system (3.11), (3.12) to the condition that ϵ is a *gauge covariantly constant* spinor in six dimensions,

$$\left(D_a + \frac{i}{2}\alpha\,(\partial_b\phi)J^b{}_a\,\gamma\right)\epsilon = 0 \quad \Leftrightarrow \quad \begin{cases}\left(D_a + \frac{i}{2}\alpha\,(\partial_b\phi)J^b{}_a\right)\epsilon_+ = 0\\ \left(D_a - \frac{i}{2}\alpha\,(\partial_b\phi)J^b{}_a\right)\epsilon_- = 0\end{cases} \tag{3.19}$$

where $\epsilon_\pm = \frac{1}{2}(1\pm\gamma)\epsilon$ are the chiral projections of ϵ. As we will show in more detail below (see eq (3.26) and the second equation in (3.30)), this last set of equations can indeed be solved on a Kähler manifold if the gauge connection cancels the U(1) part of the holonomy of the spin connection. We hasten to add that the physical metric is Kähler with respect to the $J_a{}^b$ that comes as a reduction of the octonionic structure constants; in general it wouldn't be Kähler with respect to the original complex structure of the Calabi-Yau.

In summary, we have shown that the constraints (3.6), (3.7) for $\mathcal{N} = 1$ supersymmetric compactifications of type IIA string theory to four dimensions in the presence of a non-trivial dilaton and RR two-form field strength subject to the constraint (3.18), are realized

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by a warped string frame compactification (3.2) on a non-compact Kähler manifold, provided that the dilaton and RR two-form field strength satisfy generalizations of monopole equations given in (3.14). The supersymmetry generator ϵ becomes a gauge covariantly constant spinor on the Kähler manifold. The complex structure $J_a{}^b$ and the three-form ψ_3 that appear in the construction are built using octonions. The total space of the "Kaluza-Klein bundle" has a metric (3.3) of holonomy G_2. Clearly the solution reduces to the ordinary direct Calabi-Yau compactification with constant dilaton if the RR two-form field strength is trivial, in which case it actually preserves $\mathcal{N} = 2$ supersymmetry.

Let us also add some comments about equation (3.14). In the holomorphic/antiholomorphic basis they read

$$-i\beta(\partial_c\phi) = \frac{1}{8}\check{F}^{ab}\epsilon_{abc} \qquad \Leftrightarrow \qquad \partial\left(e^{-2\beta\phi}\right) = -*\left(e^{-\alpha\phi}\psi_3^{(3,0)} \wedge F^{(2,0)}\right). \tag{3.20}$$

Equation (3.20) closely resembles the sort of "holomorphic monopole equations" described in [10, 11], where holomorphic analogues of gauge theory have been studied. The definition of "holomorphic monopole" is inspired by the expression (3.20): when written in components, it is similar to the usual monopole, but with the 3d volume ϵ_{ijk} replaced by ϵ_{abc} over holomorphic indices only. Although in our case $\epsilon_{abc}e^a e^b e^c$ (where the indices are holomorphic) is not the holomorphic three form Ω of a Calabi-Yau, but the $(3,0)$ part $\psi_3^{(3,0)}$ of our three-form ψ_3, we can still use it to define a "holomorphic Hodge dual" as in [10, 11]

$$\star: \quad \Omega^{(p,0)} \to \Omega^{(3-p,0)}, \qquad \star(\alpha) \equiv *(\bar{\alpha} \wedge \psi^{(3,0)}).$$

Using this we can rewrite (3.20) as

$$\partial\left(e^{-2\beta\phi}\right) = -e^{-\alpha\phi} \star F\,. \tag{3.21}$$

It is interesting to compare (3.14), (3.18) to the equations for gauge fields on D-branes wrapping CY manifolds [12, 8]. There, the relevant equations are known as hermitean Yang-Mills equations, for which $F^{(2,0)}$ is vanishing and $F \cdot J = \text{const}$. Note that here we have in a way an orthogonal projection, where $F^{(1,1)}$ is vanishing while equation (3.14) contains $F^{(2,0)}$ and $F^{(0,2)}$ instead. We will see that something of this sort also happens in the $\text{Spin}(7) \to G_2$ case.

In the next two subsections we will present alternative derivations of the monopole equation, using the existence of a G_2 structure and of a self-dual spin connection associated with (3.3).

3.2 Forms and monopoles: a check

From the covariantly constant spinors and the gamma matrices one can, as usual, form bilinear combinations and forms

$$J = \frac{1}{2}J_{ab}e^a e^b = -\frac{i}{2}(\epsilon^\dagger\gamma_{ab}\gamma\epsilon)\, e^a e^b\,, \qquad \psi_3 = \frac{1}{3!}\psi_{abc}e^a e^b e^c = -\frac{i}{3!}(\epsilon^\dagger\gamma_{abc}\epsilon)\, e^a e^b e^c\,. \tag{3.22}$$

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Due to the supersymmetry constraints (3.6), (3.7), these forms satisfy a set of equations, which can be found in two ways. One can simply hit them with a covariant derivative, and then use the covariant derivative for ϵ. Alternatively, one can use as a starting point the form

$$\Phi = \frac{1}{3!}\Phi_{ABC}\hat{e}^A\hat{e}^B\hat{e}^C = e^{-3\alpha\phi}\psi_3 + e^{-2\alpha\phi}J\wedge\hat{e}^z\,,$$

which defines the G_2-structure associated with the G_2 metric (3.3). More precisely, one can reduce to six dimensions the equations $d\Phi = 0 = d(*_7\Phi)$, and divide the result into pieces containing or not $\hat{e}^z$. Whatever way, one gets [4]

$$\begin{aligned} d(e^{-3\alpha\phi}\psi_3) + e^{(\beta-2\alpha)\phi}\,J\wedge F &= 0\,, & d(e^{(\beta-2\alpha)\phi}J) &= 0\,,\\ d(e^{-4\alpha\phi}(*J)) - e^{(\beta-3\alpha)\phi}\,(*\psi_3)\wedge F &= 0\,, & d(e^{(\beta-3\alpha)\phi}(*\psi_3)) &= 0\,, \end{aligned} \tag{3.23}$$

where the $*$ here denotes Hodge duality w.r.t. the physical metric ds_6^2.

Using the relation $\beta = 2\alpha$ of the ten-dimensional string frame these equations simplify considerably. First of all the second equation implies that $dJ = 0$. We stress once again that this J is $\frac{1}{2}J_{ab}e^ae^b$, the Kähler form associated to the string frame metric ds_6^2 and complex structure $J_a{}^b = -i\epsilon^\dagger\gamma_a{}^b\gamma\epsilon$. There is no contradiction here with the fact that J might happen to coincide with the Kähler form of the original Calabi-Yau metric and complex structure.

To analyze these equations further and recover the monopole equations (3.14), (3.18), we split them into bi-degrees w.r.t. the complex structure $J_a{}^b$. Using either the basis of spinors in six dimensions (see e.g. [13]) or the octonionic structure constants in an explicit basis, the two-form and three-form can be written as

$$\begin{aligned} J &= \frac{i}{2}dz^a d\bar{z}^{\bar{a}}\,,\\ \psi_3 &= \frac{1}{2}\left(\frac{1}{3!}\epsilon_{abc}dz^a dz^b dz^c + \frac{1}{3!}\epsilon_{\bar{a}\bar{b}\bar{c}}d\bar{z}^{\bar{a}}d\bar{z}^{\bar{b}}d\bar{z}^{\bar{c}}\right),\\ *\psi_3 &= \frac{i}{2}\left(\frac{1}{3!}\epsilon_{abc}dz^a dz^b dz^c - \frac{1}{3!}\epsilon_{\bar{a}\bar{b}\bar{c}}d\bar{z}^{\bar{a}}d\bar{z}^{\bar{b}}d\bar{z}^{\bar{c}}\right), \end{aligned} \tag{3.24}$$

where dz^a and $d\bar{z}^{\bar{a}}$ are frames on the holomorphic and anti-holomorphic cotangent bundle respectively. Notice that here we have suppressed the local phases $e^{\pm iq}$ of the spinors $\epsilon_\pm$ for notational convenience. Including them amounts to the replacements $\epsilon_{abc}\to e^{2iq}\epsilon_{abc}$ and $\epsilon_{\bar{a}\bar{b}\bar{c}}\to e^{-2iq}\epsilon_{\bar{a}\bar{b}\bar{c}}$ respectively.

The $(3,1)$ and $(1,3)$ parts of the first equation in (3.23) can be reduced (using also the fourth equation) to (3.14). The third equation gives instead

$$d(*J) - 4\alpha d\phi\wedge *J - \check{F}\wedge(*\psi_3) = 0\,.$$

The first term vanishes since $*J = \frac{1}{2}J\wedge J$, and the sum of the last to two terms is yet another form of the holomorphic monopole equations. For integrable complex structure the $(2,2)$-part of the first equation in (3.23) reduces to $F^{(1,1)}\wedge J = 0$, which in six dimensions is equivalent to $F^{(1,1)} = 0$. Note that in the general case however, when the complex

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structure is not integrable, $d\psi$ and $d(*\psi)$ also have a $(2,2)$ part. Both terms in the $(2,2)$ part of the first equation in (3.23) are then nonvanishing. In that case J is no longer a Kähler form, but still a symplectic form.

To get equation (3.19) back from (3.23) is less direct, the forms being bilinear in the spinor. Here we content ourselves with reformulating the fourth equation in (3.23) in a more suggestive way. Using a holomorphic basis for the vielbein and the hermiticity of the metric, we can define the quantity $e_h = \frac{1}{6}\epsilon_{abc}\epsilon^{ijk}e^a_i e^b_j e^c_k$, a sort of holomorphic part of the determinant of the metric, such that $\sqrt{|g|} = e_h \bar{e}_h$. Then splitting the fourth equation in (3.23) gives

$$d\left(e^{-\alpha\phi}(*\psi_3)\right) = 0 \quad \Leftrightarrow \quad \partial\left(e^{-\alpha\phi}\psi_3^{(0,3)}\right) = 0 = \overline{\partial}\left(e^{-\alpha\phi}\psi_3^{(3,0)}\right) \quad \Leftrightarrow \quad d\left(e^{-\alpha\phi}\psi_3\right) = 0. \tag{3.25}$$

In particular this implies

$$\bar{\partial}\left(e^{-\alpha\phi}e_h\right) = 0\,. \tag{3.26}$$

We can actually use the equations (3.23) to characterize the sub-manifold M which the N supersymmetric D6-branes wrap. Being a magnetic source of charge N for F, one has $dF = N\delta_M$, where δ_M is the Poincaré dual three-form of the cycle M. Taking the exterior derivative of the first and third equation of (3.23) and using (3.25) one finds that

$$J \wedge \delta_M = 0\,, \qquad e^{-\alpha\phi}(*\psi_3) \wedge \delta_M = 0\,. \tag{3.27}$$

Moreover, the monopole equation (3.14) implies

$$\Delta\left(e^{-2\beta\phi}\right) = -N * \left(e^{-\alpha\phi}\psi_3 \wedge \delta_M\right). \tag{3.28}$$

The first equation in (3.27) means that M is a lagrangian cycle in the internal Kähler manifold. The second equation in (3.27) together with (3.28) are what the additional condition of being a *special* lagrangian cycle w.r.t. the Calabi-Yau structure turns into after the back reaction of the branes on the physical metric is taken into account.

3.3 Monopoles and self-dual spin connections

As in the case of the three-dimensional monopole equation (2.1), the monopole equations (3.13), (3.14) can be traced back to the self-duality of the spin connection in the lift to one dimension higher.

As shown in [5], the SO(7)-gauge freedom of the spin connection $\hat{\omega}_{AC} = \hat{\Gamma}_{ABC}\hat{e}^B$ associated with the G_2 metric ds_7^2 can be used to make the latter self-dual, in the sense that

$$\hat{\omega}_{AB} = \frac{1}{2}(*\Phi)_{ABCD}\hat{\omega}^{CD} \qquad \Leftrightarrow \qquad \Phi_{ABC}\hat{\omega}^{AB} = 0\,. \tag{3.29}$$

Furthermore, this is the gauge choice for which the covariantly constant spinor ϵ is actually constant and has a single nonvanishing entry, namely the singlet in $\underline{8} \to \underline{7} + \underline{1}$.

Equation (3.29) implies

$$0 = \Phi_{ab7}\hat{\omega}^{ab} = J_{ab}\left(\omega^{ab} + 2\alpha(\partial^a\phi)e^b - \frac{1}{2}e^{\alpha\phi}\check{F}^{ab}\hat{e}^z\right)$$

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and

$$
\begin{aligned}
0 &= \Phi_{ABc}\hat{\omega}^{AB} = \psi_{abc}\hat{\omega}^{ab} + 2J_{bc}\hat{\omega}^{7b} \\
&= \psi_{abc}\left(\omega^{ab} + 2\alpha(\partial^a\phi)e^b - \frac{1}{2}e^{\alpha\phi}\check{F}^{ab}\hat{e}^z\right) + 2J_{bc}\left(\frac{1}{2}\check{F}^b{}_d e^d + \beta e^{\alpha\phi}(\partial^b\phi)\hat{e}^z\right).
\end{aligned}
$$

Decomposing these equations into the basis $e^a, \hat{e}^7$ we find the following constraints

$$
\begin{aligned}
0 &= J_{ab}F^{ab}\,, \\
0 &= J_{ab}\left(\Gamma^a{}_c{}^b + 2\alpha\delta_c{}^b(\partial^a\phi)\right), \\
0 &= \beta(\partial^b\phi)J_{bc} - \frac{1}{4}\check{F}^{ab}\psi_{abc}\,, \\
0 &= \check{F}_d{}^b J_{bc} - \left(\Gamma^a{}_d{}^b + 2\alpha\delta_d{}^b(\partial^a\phi)\right)\psi_{abc}\,.
\end{aligned}
\tag{3.30}
$$

The same equations also follow from the type IIA supersymmetry constraints (3.6), (3.7). We recall that in the SO(7)-gauge in which the spin connection $\hat{\omega}_{AB}$ is self-dual, the spinor ϵ has a single constant component. In this gauge the derivative term in (3.6) drops out, and, using (3.9), we can rewrite (3.6), (3.7) as

$$
0 = \left\{\left[\left(\Gamma^a{}_d{}^b + 2\alpha\delta_d{}^b(\partial^a\phi)\right)\psi_{abc} - \check{F}_d{}^b J_{bc}\right]\gamma^c + \left[\left(\Gamma^a{}_d{}^b + 2\alpha\delta_d{}^b(\partial^a\phi)\right)J_{ab}\right]\gamma\right\}\epsilon \tag{3.31}
$$

and

$$
0 = \left\{\left[-\beta(\partial^b\phi)J_{bc} + \frac{1}{4}\check{F}^{ab}\psi_{abc}\right]\gamma^c + \left[\frac{1}{4}\check{F}^{ab}J_{ab}\right]\gamma\right\}\epsilon\,. \tag{3.32}
$$

Decomposing into a basis $\epsilon, \gamma^A\epsilon$ for the spinors, we obtain the same constraints (3.30) as by using the self-duality of the spin connection.

The first and third equation in (3.30) give once again the monopole equations (3.13), (3.14). Inserting the third equation into the fourth and using the string frame relation $\beta = 2\alpha$ one derives the constraint

$$
\Gamma^a{}_d{}^b\psi_{abc} = \frac{1}{2}\left(\check{F}_{cb}J^b{}_d + \check{F}_{db}J^b{}_c\right). \tag{3.33}
$$

Under our assumption $F^{(1,1)} = 0$, (3.18), the right hand side is identically zero. $\Gamma^a{}_d{}^b\psi_{abc} = 0$ then implies the projection of the general SO(6) holonomy of the spin connection $\underline{15} = \underline{9} + \underline{3} + \bar{\underline{3}} \mapsto \underline{9}$ into U(3) holonomy. Hence the geometry is Kähler.

Finally, the second equation in (3.30) implies that $2\alpha(\partial^a\phi)J_{ac}$ defines a U(1) gauge connection for the six-dimensional spinor that cancels the U(1) part of the holonomy of the spin connection. Indeed, the term $J_{ab}\Gamma^a{}_d{}^b$ is the U(1) part of the connection. Due to Kählerity, the connection has a particularly compact expression, that allows to rewrite the second equation as

$$
E^{\bar{a}k}\bar{\partial}_{\bar{j}}e^a{}_k = \alpha\bar{\partial}_{\bar{j}}\phi\,,
$$

where E is the inverse vielbein. Now, the first piece is nothing but $TrE^{-1}\bar{\partial}_{\bar{j}}e = \bar{\partial}_{\bar{j}}(\log e_h)$; so we get $\bar{\partial}_{\bar{j}}(\log e_h - \alpha\phi) = 0$, which is (3.26). Finally, we are now able to show more directly the connection between (3.19) and the Kählerity condition. As we have just done for (3.31), use the action of the gamma matrices on the spinor and gauge the latter to be a constant: it is easy to see that Kählerity follows, in the form $\Gamma^a{}_d{}^b\psi_{abc} = 0$, and one is left with the second equation in (3.30).

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4. Spin(7) → G_2

This case will not be treated in as much detail as the previous one; we will only show that a "G_2 monopole equation" arises. The equation in seven dimensions that one gets from the reduction of eight dimensions is formally the same as in (3.6), (3.7), but for the fact that γ is now the identity and can be dropped, ϵ being now a spinor in seven dimensions which was a Weyl spinor in eight dimensions. A basis for spinors is then spanned by ϵ and $\gamma^A\epsilon$ [8]; moreover we have

$$\gamma_{AB}\epsilon = i\Phi_{ABC}\gamma^C\epsilon\,. \tag{4.1}$$

Using this, the gravitino and dilatino equations become

$$D_A\epsilon + i\left[\frac{\alpha}{2}(\partial^B\phi)\,\Phi_{BAC} + \frac{1}{4}\check{F}_{AC}\right]\gamma^C\epsilon = 0\,. \tag{4.2}$$

$$i\left[\frac{1}{4}\check{F}^{AB}\,\Phi_{ABC} + \beta\,\partial_C\phi\right]\gamma^C\epsilon = 0\,. \tag{4.3}$$

The dilatino equation gives us again a monopole equation. Reinserting this equation back into the gravitino one, the latter becomes

$$D_A\epsilon = -\frac{i}{4}\left[\left(1-\frac{\alpha}{\beta}\right)\check{F}_{AC} + \frac{\alpha}{2\beta}(*\Phi)_{ACDE}\,\check{F}^{DE}\right]\gamma^C\epsilon\,. \tag{4.4}$$

In a frame with $\beta = 3\alpha$ the right hand side vanishes if

$$\check{F}_{AB} + \frac{1}{4}(*\Phi)_{ABCD}\,\check{F}^{CD} = 0\,. \tag{4.5}$$

The last condition means that in the reduction $\underline{21} \to \underline{14}+\underline{7}$ of the adjoint of SO(7) under its subgroup G_2, we project out the fields in the $\underline{14}$, whereas for G_2 instantons it is the $\underline{7}$ that is projected out [14]. This condition is the analogue of the projection (3.18) in six dimensions, where F was assumed to have components only in the representations $\underline{3}+\underline{\bar{3}}$, orthogonal to what one has for hermitean Yang-Mills equations (see the remark after (3.20)).

Assuming (4.5) to hold, the gravitino equation requires the physical metric in the frame $\beta = 3\alpha$ to allow for a covariantly constant spinor and hence to be of holonomy G_2. This in turn implies that the physical string frame metric, for which $\beta = 2\alpha$, is conformal to a G_2 metric,

$$ds_7^2 = e^{\frac{2}{3}\alpha\phi}ds^2_{G_2}\,. \tag{4.6}$$

Analogously to the subsection 3.3, the same conclusions can again be drawn from the existence of a self-dual spin connection on the eight-dimensional lift. In that gauge the spinor is constant. Whereas the dilatino equation (4.3) stays unchanged, the gravitino equation, after reinserting the dilatino one, becomes

$$\Gamma^B{}_A{}^D\Phi_{BDC} = -\left[\left(1-\frac{\alpha}{\beta}\right)\check{F}_{AC} + \frac{\alpha}{2\beta}(*\Phi)_{ACDE}\,\check{F}^{DE}\right]. \tag{4.7}$$

Assuming (4.5) one sees that the spin connection coefficients in the frame $\beta = 3\alpha$ satisfy the self-duality condition $\Gamma^B{}_A{}^D\Phi_{BDC} = 0$, so that the physical metric in this frame has G_2 holonomy.

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It is also not difficult to derive the analogue of (3.23) from the fact that the lift to eight dimensions has a Spin(7)-structure,

$$\Omega_8 = -e^{-4\alpha\phi}(*\Phi) - e^{-3\alpha\phi}\Phi \wedge \hat{e}^z \,. \tag{4.8}$$

Reducing $d\Omega_8 = 0$ to seven dimensions one gets

$$d(e^{-4\alpha\phi}\,(*\Phi)) = e^{(-3\alpha+\beta)\phi}\,\Phi \wedge F\,, \qquad d(e^{(-3\alpha+\beta)\phi}\Phi) = 0\,. \tag{4.9}$$

Again, if we suppose $3\alpha = \beta$, the second equation yields $d\Phi = 0$. The first one can be rewritten as

$$d(*\Phi) - 4\alpha d\phi \wedge *\Phi - \check{F} \wedge \Phi = 0\,,$$

and the most natural way to solve it is to set the sum of the last two terms to zero, which gives our monopole equation coming from the dilatino, (4.3), and $d(*\Phi) = 0$. The latter together with $d\Phi = 0$ implies once again that the metric in the frame $\beta = 3\alpha$ has a holonomy contained in G_2.

From (4.9) we can again characterize the submanifold M on which F has a source, $dF = N\delta_M$. Hitting the first equation with d and using the second, we obtain the condition

$$e^{(-3\alpha+\beta)\phi}\Phi \wedge \delta_M = 0\,,$$

which means that the cycle is coassociative with respect to the G_2 structure Φ in the frame $\beta = 3\alpha$. Likewise, the monopole equation (4.3) can be rewritten as $d(e^{-4\alpha\phi}) = \frac{2}{3} * (F \wedge *\Phi)$, or

$$\Delta\,(e^{-4\alpha\phi}) = \frac{2}{3}N * (\delta_M \wedge *\Phi)\,. \tag{4.10}$$

5. Conclusion

We have seen how the conditions for preserving supersymmetry in type IIA string theory in the presence of a nontrivial dilaton and RR two-form field strength get reduced to generalizations of the monopole equations. The essential ingredient of this generalization is the existence of a three-form in the underlying geometry. The special role played by three-forms in describing six- and seven-dimensional structures is well-studied [15]. Here our emphasis has been on the circle fibrations on such spaces, and we have shown that in order for the total space of such a "Kaluza-Klein circle bundle" to yield a manifold of restricted holonomy, the volume of the circle, the field strength of the Kaluza-Klein vector and the three-form have to satisfy generalized monopole equations. If moreover the field strength has additional projection properties, (3.18) or (4.5), the base manifolds with their string frame metric are Kähler for $d = 6$ and conformal to a G_2 manifold for $d = 7$.

It would be desirable to characterize the base geometries further, even if these supplementary constraints are not imposed. This even more, since they prohibit an easy inclusion of the cases with lower d into those for higher d, as can be done for vanishing background fluxes, where we have successive inclusions of the holonomies, $\{1\} \subset \mathrm{SU}(3) \subset \mathrm{G}_2$ and $\mathrm{SU}(2) \subset \mathrm{G}_2 \subset \mathrm{Spin}(7)$, respectively. For example the known solution for D6-branes in flat

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space, described in section 2, satisfies the conditions (3.6), (3.7) for $\mathcal{N} = 1$ supersymmetry in four dimensions, or alternatively their form (3.13)–(3.15). However, w.r.t. the almost complex structure that descends from the octonions, $F^{(1,1)}$ is nonvanishing. This almost complex structure is therefore not integrable and the string frame metric on the internal six-dimensional space not Kähler.

A better understanding of the general base geometries and the conditions on the connections on the circle bundles should hopefully lead to some systematic construction of new metrics of restricted holonomy. One may apply a warping $ds_6^2 = e^{\beta\phi} ds_3 + e^{-\beta\phi} d\tilde{s}_3$, analogous to the flat case, to a cotangent bundle of any three-dimensional manifold. The latter are local realization of SLAGs inside a Calabi-Yau threefold. If one is able to solve (3.13),–(3.15) for ϕ and A, one could get at least local expressions for new metrics of G_2 holonomy.

We would like to conclude this paper by mentioning another direction for further research that we have not pursued here. The possibility of having more general setups where other RR fluxes are turned on is of obvious physical interest. In some cases this lifts to internal manifolds of weak restricted holonomy, on which spinors instead of being covariantly constant satisfy $D_A \epsilon = i \lambda \gamma_A \epsilon$. Further deformations of both the underlying geometry and the monopole equations arising in this context have been left for future work.

Acknowledgments

We would like to thank B. Acharya, D. Calderbank, M. Douglas, D. Martelli for useful discussions. This work is supported in part by EU contract HPRN-CT-2000-00122 and by INTAS contracts 55-1-590 and 00-0334. PK and MP are supported by European Commission Marie Curie Postdoctoral Fellowships under contract numbers HPMF-CT-2000-00919 and HPMF-CT-2001-01277.

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